

COMBINATORIAL REASONING

Combinatorial Argument.

Definition

A partition of a set A is a collection of pairwise disjoint subsets of A whose union is A ,

Example A is finite set and a_1, a_2, \dots, a_n is partition of set A . Then, by **rule of sum**,

$$|A| = \sum_{i=1}^m |a_i|$$

Definition

Let T be a set whose each elements of T can be described using a procedure involving step S_1, S_2, \dots, S_k . There are r_i ways to do each step. Then,

$$|T| = \prod_{i=1}^m r_i$$

Definition

Permutation of finite set S is a bijection from set S to itself, defined as $f : S \rightarrow S$

Theorem 01. *A set of n element has $n!$ permutation. In general number of arrangement of k distinct element is*

$$P(n, k) = \frac{n!}{(n-k)!} = n(n-1)\dots(n-(k-1))$$

Proof. Consider an injection function $f : [k] \rightarrow S$ where $|S| = n$ and $[k] = \{1, 2, \dots, k\}$. We count the number of injections from $[k]$ to S by sequentially choosing the image of each element in $[k]$. For first position ($i = 1$), we have n options to choose from set S , once we have chosen our first position, we have $n - 1$ remaining elements to choose for second position. In general, for i th position, we have $n - (i - 1)$ choices remaining. It follows that $n(n-1)\dots(n-k+1)$ \square

Definition

A selection of k element subset of $[n]$ is $\binom{n}{k}$

Theorem 02. *For integers n, k with $0 \leq k \leq n$,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. We will prove this by using arrangements. In Theorem 1, permutation does consider order so it will overcount k arrangements. In permutations, the order of the

selected elements matters, leading to overcounting when we are only interested in selections (where order does not matter). Specifically, each selection of k elements is counted $k!$ times, corresponding to the $k!$ possible arrangements of those k elements.

To obtain the number of combinations, we need to eliminate this overcounting. Therefore, we divide the number of permutations by $k!$. We have

$$C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{(n-k)!k!}$$

□

Theorem 03. (*Binomial Theorem*)

$$(x + y)^n = (x + y)(x + y)\dots(x + y) = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Proof. The left hand side is multiplying factor n times. The right hand side is we must choose x and y from each factor. To form $x^k y^{n-k}$, we must select x from k of the factor n and y from remaining $n - k$ factors. The number of ways to choose this k is given by $\binom{n}{k}$. Summing over all possible k from 0 to n , gives

$$(x + y)^n = (x + y)(x + y)\dots(x + y) = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

□

Lemma

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof. LHS, we choose subsets of k elements from n elements. When we chose k elements, we leave out $n - k$ elements. Therefore, this is equivalent as choosing subsets of $n - k$ elements from n elements (RHS). □

Lemma (Pascal Formula)

If $n \geq 1$, then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof. We want to choose subset of size k element from n element. We can do it by partition our set. Suppose we have set $A = \{a_1, \dots, a_n\}$. Let's partition our set into set $A_1 = \{a_1, \dots, a_{n-1}\}$ and set $A_2 = \{a_n\}$ element. Let's consider 2 disjoint cases.

- subset of k element contains a_n
we only have $k - 1$ element subset to choose from $n - 1$ element, that is

$$\binom{n-1}{k-1}$$

- subset of k element contains a_n
then we have k element subset to choose from $n - 1$ element, that is

$$\binom{n-1}{k}$$

Since these two cases are disjoint and cover all possible subsets of size k , the total number of subsets of size k from A is the sum of the counts from each case:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

□

Theorem 04. *With repetition allowed, there are*

$$\binom{n+k-1}{k-1}$$

to select n objects from k types.

Proof. We use stars and bars method to show this. Consider n stars and $k-1$ bars. Let's this $k-1$ bars divide n objects into k types. Suppose we have $n=3$ and $k=3$. Then, we have $**|*|$ which represent 3 types, that is 2 objects in type 1 and 1 object in type 2 and 0 object in type 3. Since repetition is allowed, some type may have 0 objects. The total number of stars and bars is $n+k-1$ and the position of this separating bars that we need to choose is $k-1$ □

Lemma (The Chairperson Identity)

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Proof. Consider a group of n people, and we want to form a committee of k people and then choose a chairperson from that committee.

LHS: First, we choose a committee of k people from the n people, which can be done in $\binom{n}{k}$ ways. Then, from this chosen committee, we select a chairperson, which can be done in $\binom{k}{1} = k$ ways. Therefore, the total number of ways to form the committee and choose a chairperson is $k\binom{n}{k}$.

RHS: Alternatively, we can first choose the chairperson from the n people, which can be done in $\binom{n}{1} = n$ ways. Then, we choose the remaining $k-1$ members of the committee from the remaining $n-1$ people, which can be done in $\binom{n-1}{k-1}$ ways. Therefore, the total number of ways to form the committee and choose a chairperson is $n\binom{n-1}{k-1}$.

Since both the LHS and RHS count the same thing, we have:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

□

Lemma (The Summation Identity)

$$\sum_{i=0}^n \binom{i}{k} = \binom{n+1}{k+1}$$

Proof. Consider binary tuples of length $n + 1$ with exactly $k + 1$ ones. We want to count the number of such tuples.

LHS: Let's count the tuple based on the position of rightmost 1. The position is numbered from 1 to $n + 1$, that is $1, 2, \dots, n + 1$. Let's the position of rightmost 1 is $i + 1$ (1.....100), then the remaining k ones must be positioned in first i position. The number of ways to do this is $\binom{i}{k}$. Since the rightmost 1 position can be anywhere between $k + 1$ to $n + 1$ (1...1000), summing all over i , gives us $\sum_{i=0}^n \binom{i}{k}$

RHS: The total of binary tuple is $n + 1$ with $k + 1$ ones. That is $\binom{n+1}{k+1}$

Since both LHS and RHS count the same thing, it completes the proof. □

Example Comparison of poker hands.

We have 52 cards with 13 ranks and 4 suits.

- choose 5 cards at random

$$\binom{52}{5} = 2,598,960$$

- choose 3 cards of the same rank and one in each of two other ranks

$$\binom{13}{1} \binom{4}{3} \binom{13}{2} \binom{4}{1} \binom{4}{1}$$

- 2 cards in two ranks and fifth card in some third rank

$$\binom{13}{2} \binom{4}{2} \binom{4}{2} \binom{44}{1}$$